Correction Model Midterm Exam

Linear Algebra 2, March 2, 2020

1. a) For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ we have $tr(A^T B) = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

<u>Axiom 1</u>: $\langle A, A\rangle = tr(A^T A)$

(2) $\qquad = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \geq 0$

In addition: $\langle A, A\rangle = 0 \iff a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 = 0$

(1) $\qquad \iff a_{11} = a_{12} = a_{21} = a_{22} = 0$

$\qquad \iff A = 0$

<u>Axiom 2</u> $\langle B, A\rangle = b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + b_{22}a_{22}$

(1) $\qquad = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

$\qquad = \langle A, B\rangle.$

<u>Axiom 3</u>

$\langle \alpha A + \beta B, C\rangle =$

$(\alpha a_{11} + \beta b_{11})c_{11} + (\alpha a_{12} + \beta b_{12})c_{12}$

(2) $+ (\alpha a_{21} + \beta b_{21})c_{21} + (\alpha a_{22} + \beta b_{22})c_{22}$

$= \alpha(a_{11}c_{11} + a_{12}c_{12} + a_{21}c_{21} + a_{22}c_{22})$

$+ \beta(b_{11}c_{11} + b_{12}c_{12} + b_{21}c_{21} + b_{22}c_{22})$

$= \alpha\langle A, C\rangle + \beta\langle B, C\rangle.$

b) $S = \{ \alpha M_1 + \beta M_2 \mid \alpha, \beta \in \mathbb{R} \}$

$$= \left\{ \begin{pmatrix} \beta & 0 \\ 0 & \alpha+\beta \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}.$$

Note that S has dimension 2 : it is spanned by the 2 linearly independent matrices M_1 and M_2.

Define
$$E_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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Clearly $\langle E_1, E_2 \rangle = 0$ and $\| E_1 \| = 1$, $\| E_2 \| = 1$. Hence $\{E_1, E_2\}$ is an orthonormal set.

Also $E_2 = M_2 \in S$ and $E_1 = M_2 - M_1 \in S$

We conclude that $\{E_1, E_2\}$ is an orthormal basis of S, since $\mathrm{Span}(E_1, E_2) = S$

c) $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The least squares approximation is equal to the projection P of M onto the subspace S :

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$$P = \langle M, E_1 \rangle \cdot E_1 + \langle M, E_2 \rangle \cdot E_2$$
$$= a \cdot E_1 + d\, E_2 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

2) a) Note that for vectors $\tilde{p}, \tilde{q} \in \mathbb{R}^n$ we have that $\tilde{p}^T \tilde{q}$ is the scalar product, which is an inner product. We will use this fact.

<u>Axiom 1</u> $\langle p(x), p(x) \rangle = \tilde{p}^T \tilde{p} = \|\tilde{p}\|^2 \geq 0$

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In addition $\langle p(x), p(x) \rangle = 0 \iff \|\tilde{p}\|^2 = 0 \iff$
$\tilde{p} = 0 \iff p(c_1) = p(c_2) = \dots = p(c_n) = 0$

Hence the polynomial $p(x)$ of degree $\leq n-1$ has n <u>distinct</u> roots. This can only occur if the polynomial $p(x)$ itself is the zero polynomial $p(x) = 0$.

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<u>Axiom 2</u> $\langle p(x), q(x) \rangle = \tilde{p}^T \tilde{q} = \tilde{q}^T \tilde{p}$
$= \langle q(x), p(x) \rangle$

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<u>Axiom 3</u> $\langle \alpha p(x) + \beta q(x), r(x) \rangle =$

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$(\alpha \tilde{p} + \beta \tilde{q})^T \tilde{r} = \alpha \tilde{p}^T \tilde{r} + \beta \tilde{q}^T \tilde{r}$
$= \alpha \langle p(x), r(x) \rangle + \beta \langle q(x), r(x) \rangle$.

b) $\tilde{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\tilde{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

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Hence $\langle 1, x \rangle = \tilde{1}^T \tilde{x} = 0$

c) P_2 consists of all polynomials of degree ≤ 1
This subspace has the basis $\{1, x\}$
We already saw that 1 and x are orthogonal
We normalize them: $\|1\| = (\tilde{1}^T \tilde{1})^{1/2} = \sqrt{3}$

4 and $\|x\| = (\tilde{x}^T \tilde{x})^{1/2} = \sqrt{2}$

Hence, an orthonormal basis of P_2 is given by $\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x\}$.

d) Let $p(x)$ be the orthogonal projection of x^2 onto P_2. Then

$$p(x) = \langle x^2, \tfrac{1}{\sqrt{3}} \rangle \cdot \tfrac{1}{\sqrt{3}} + \langle x^2, \tfrac{1}{\sqrt{2}}x \rangle \tfrac{1}{\sqrt{2}}x$$

We compute:

$$\langle x^2, \tfrac{1}{\sqrt{3}} \rangle = (1 \ 0 \ 1) \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{2}{\sqrt{3}}$$

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$$\langle x^2, \tfrac{1}{\sqrt{2}}x \rangle = (1 \ 0 \ 1) \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

Hence $p(x) = \frac{2}{3}$

3) a) $M^H = (A + iA^T)^H = (\overline{A + iA^T})^T$

5 $(A - iA^T)^T = A^T - iA = -i(A + iA^T)$

$= -iM$

b) Since $M \in \mathbb{C}^{n \times n}$ the <u>Schur decomposition</u> can be applied: there exist a <u>unitary</u> matrix U and an <u>upper triangular</u> matrix T such that $U^H M U = T$.

Now observe that

$$T^H = (U^H M U)^H$$
$$= U^H M^H U$$
$$= U^H(-iM)U$$
$$= -i\, U^H M U = -iT,$$

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so $T^H = -iT$. This says that the upper triangular matrix T must, in fact, be a <u>diagonal</u> <u>matrix</u> !

c) The diagonal elements of T are the eigenvalues of M. According to b) these diagonal elements λ satisfy $\bar{\lambda} = -i\lambda$. Put $\lambda = x+iy$. Then $\bar{\lambda} = x-iy$ and $-i\lambda = y-ix$.

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Thus $\bar{\lambda} = -i\lambda \Leftrightarrow x=y$.

We conclude that $\lambda \in \{x+ix \mid x \in \mathbb{R}\}$

4) a) $(A^H A)^H = A^H (A^H)^H = A^H A$

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b) Let $\lambda \neq 0$ be an eigenvalue of $A^H A$. Then there exists $x \neq 0$ such that $A^H A x = \lambda x$.

This implies $AA^HAx = \lambda Ax$

Put $z := Ax$. Then $z \neq 0$ since

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Also $AA^Hz = \lambda z$, so λ is also an eigenvalue of AA^H.

Conversely, if $\lambda \neq 0$ eigenvalue of AA^H then it is an eigenvalue of A^HA. This follows in the same way

c) There exists $x_i \neq 0$ such that $A^HAx_i = \lambda_i x_i$

This implies $x_i^H A^H A x_i = \lambda_i x_i^H x_i$, which is equivalent with

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$$\|Ax_i\|^2 = \lambda_i \|x_i\|^2.$$

Hence
$$\lambda_i = \frac{\|Ax_i\|^2}{\|x_i\|^2} \geq 0$$

d) We already know $\lambda_i \geq 0$ for all i

Assume there exists i such that $\lambda_i = 0$

Then there exists a corresponding

6 eigenvector $x_i \neq 0$ such that

$Ax_i = \lambda_i x_i = 0$, so $x_i \in N(A)$.

We conclude that A has linearly dependent columns.